Source of relaxation in the one-dimensional gravitating system

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Recent numerical experiments suggest that the one-dimensional system consisting of N parallel mass sheets relaxes on two time scales: (1) a rapid violent phase with duration of order t_c (a typical system crossing time) resulting in a stationary quasiequilibrium distribution, followed by (2) a gradual succession of quasiequilibrium states leading finally to thermal equilibrium. Each quasiequilibrium state is characterized by a stationary solution of the Vlasov equation within which fluctuations relax on a (microscopic) time scale on the order of $\alpha N t_c$ with $\alpha \approx 1$, while the final macroscopic evolution of quasiequilibrium states takes place in $\beta N t_c$, with $\beta \approx 10^4$. The purpose of this paper is to demonstrate that both the microscopic and macroscopic relaxation time scales can be completely explained within the context of the diffusion model developed specifically for this system. [S1063-651X(96)51905-7]

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I. INTRODUCTION

The system of parallel mass sheets (SPMSs) has proved to be popular for exploring the evolution of a classical N body system in which only gravitational forces act. The mass sheets are of constant density and infinite extent and move in one dimension perpendicular to their surface. Since, between crossings, each mass sheet has constant acceleration, the system evolution can be simulated on a computer without resorting to the stepwise integration of coupled differential equations. Thus it is possible to evolve the system rapidly and with high precision for very long time periods and, consequently, it was possible to investigate many of the system's dynamical features even with the slower technology available in the previous two decades.

Astronomers and astrophysicists have viewed the system as an idealized one-dimensional ''galaxy'' and used it to test various theories of relaxation such as Lynden-Bell's predictions [1] for the final state of the initial violent phase [2] or, more recently, Schmidt and Weichen's variational theory of Vlasov relaxation [3,4]. It has even been conjectured that the system represents the motion of stars perpendicular to the plane of a highly flattened galaxy [5]. However, as the simplest N body gravitational system, the model merits study in its own right.

Some time ago, by observing the evolution of small N systems, Hohl and Feix [6] conjectured that the SPMS relaxes to the state of thermal equilibrium on a time scale of $N^2 t_c$. This estimate was accepted by the astronomy community and reported in the literature for over a decade [7]. Careful study of systems consisting of 100 sheets carried out by Wright *et al.* did not support their conclusions [8] and led to a number of inconclusive studies concerning the system's ergodic properties and the eventual outcome of the evolutionary process [9,10]. However, it was apparent from all of these studies that thermalization, if it occurred, was a very slow process. Reidl and Miller [11] demonstrated that a class of stable orbits became unstable for $N > 10$. By extrapolating the convergence of the Lyapunov exponents from distinct unstable regions of the phase space they conjectured that, for $N \geqslant 11$, the system was ergodic and thermalized on a time scale of at least $10^7 t_c$. Shortly thereafter, Tsuchiya *et al.* [12] confirmed the prediction by direct simulation. Very recently they demonstrated [13] that the macroscopic thermalization time scale seems to go as $\beta N t_c$, with $\beta \approx 10^4$. They also observed that the relaxation of microscopic fluctuations occurred similarly in $\alpha N t_c$ but with $\alpha \approx 1$ [13].

In a parallel work, Miller developed an *ab initio* theory for the relaxation of fluctuations about equilibrium in the SPMS under conditions of large N [14]. It was developed from the single assumption that the acceleration and velocity of an individual sheet is approximately Markovian. Very recently Yawn and Miller extended the approach [15] to include microscopic relaxation in an arbitrary quasiequilibrium state, where, by quasiequilibrium, I mean a μ space (position, velocity) distribution which is stationary under a ''collisionless'' Vlasov evolution [16].

The diffusion theory is based on the observation that when a pair of mass sheets cross, the change in acceleration of each sheet is discontinuous and of the order $1/N$. Thus, in SPMSs, acceleration plays the role of the velocity of the proverbial ''pollen grain'' in the classic theory of Brownian motion [17]. Under the assumption that sheet crossings occur at random times, it was shown that the usual, deterministic, Vlasov evolution was recovered in the limit of large N. In addition, it was shown that by scaling the fluctuations in acceleration and velocity by $\sqrt{N}$, a Fokker-Planck equation for the evolution of the probability distribution of the *scaled* fluctuations in acceleration and velocity emerges from the $N \rightarrow \infty$ limit and defines a generalized Ornstein-Uhlenbeck process [18].

In the following paragraphs I will show that the Markov approach can be employed to obtain a diffusion equation for the evolution of the energy distribution of a mass sheet. The theory proceeds by first observing that all of the stationary Vlasov orbits are periodic, and then demonstrating that the variance in energy in a single Vlasov orbit is independent of the initial phase. Under the Markov assumption, it is shown that the variance in energy per cycle decreases exactly as $1/N$. With the additional assumption that the energy of a mass sheet is Markovian on the intermediate time scale of

the orbital period, an alternative Fokker-Planck equation for the evolution of the energy distribution is obtained. This orbit averaged evolution equation can be applied directly to the problem of microscopic relaxation. It can also be adapted to macroscopic relaxation in which the quasiequilibrium state evolves very slowly. In each case, since the variance in energy per cycle scales exactly as $1/N$, the relaxation time for the appropriate process must be strictly proportional to N. As the numerical experiments demonstrate, the two relaxation time scales are distinguished only by the coefficient of N [13].

Here I will first briefly review the Markov theory. I will then derive a differential equation for the variance of the scaled energy of a mass sheet and use it to show (1) that the variance per cycle is independent of the initial orbit phase, and (2) that it scales exactly as $1/N$. For the specific case of equilibrium the numerical solution for the variance will be examined and it will be apparent that it has all of the claimed properties. This leads naturally to the Fokker-Planck equation for the evolution of the energy distribution. Finally, I will indicate how solutions can be obtained in principle for each type of relaxation.

II. THE *A-V* PROCESS

First I will describe the setup [15]. I have chosen units where both the total sheet surface density and $2\pi G$ (G is the gravitational constant) are unity. Let x_j, v_j, and a_j denote, respectively, the position, velocity, and acceleration of mass sheet j; $1 \leqslant j \leqslant N$. The acceleration of a sheet is proportional to the difference between the mass density on its right and left. In these units $a_j = (N-2j+1)/N$ [15], where here the sheet labels are ordered from left to right. Thus the system of sheets can be regarded as N particles of mass $m=1/N$ moving on the line. Each particle experiences a constant acceleration until it encounters another. When two particles cross they simply pass through each other and exchange acceleration. The energy of this one-dimensional conservative system is simply

$$E = \frac{1}{2N}\sum_{j=1}^{N} v_j^2 + \frac{1}{N^2}\sum_{1\leqslant i<j\leqslant N} |x_i - x_j|. \tag{1}$$

Let $f_N(x,v)$ denote the singlet distribution in μ space (position, velocity), but normalized to unity, and $\rho_N(x)$ denote the density as

$$f_N(x,v) = \left\langle \frac{1}{N}\sum_{i=1}^{N} \delta(x-x_i)\,\delta(v-v_i) \right\rangle, \tag{2}$$

$$\rho_N(x,v) = \left\langle \frac{1}{N}\sum_{i=1}^{N} \delta(x-x_i) \right\rangle = \int_{-\infty}^{\infty} f_N(x,v)\,dv, \tag{3}$$

where the angle brackets denote the ensemble average. Then, with these units, $f_N(x,v)$ is also the mass density in μ space. In thermal equilibrium, Rybicki showed that [19]

$$\lim_{N\to\infty} f_N(x,v) = \frac{1}{2\sqrt{\pi}} e^{-v^2} \operatorname{sech}^2 x. \tag{4}$$

In the $N\to\infty$ limit, the system can be regarded as a continuous fluid with μ space density equal to $f(x,v)$ which evolves under the collisionless Vlasov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial v} = 0, \tag{5}$$

where $a = a(x,[\rho])$ is the acceleration of a fluid mass point and depends functionally on the density ρ:

$$a(x,[\rho]) = \int_x^{\infty} dx'\,\rho(x') - \int_{-\infty}^{x} dx'\,\rho(x') = -\frac{\partial \phi}{\partial x} \tag{6}$$

and $\phi(x)$ is the local potential function associated with f. In a stable, stationary, Vlasov flow each mass point executes periodic motion with constant energy per unit mass $\varepsilon = (\frac{1}{2})v^2 + \phi(x)$ and period $T(\varepsilon)$.

For sufficiently large N, the actual motion of a particle in the SPMS closely approximates that of a fluid mass point following the Vlasov flow. However, as time progresses, the accumulation of discreteness effects resulting from random crossings with its neighbors cause the particle to drift away from the original Vlasov orbit. Thus, when N is large, particle motion can be regarded as the slow sampling of a sequence of nearby Vlasov trajectories. Numerous simulations have demonstrated that, following the initial relaxation period, the system settles down in a quasiequilibrium state which changes very slowly [8,10]. Thus it can be assumed that, subsequent to this initial period, $f_N(x,v)$ is closely represented by a stationary Vlasov distribution, say $f(x,v)$, which evolves very slowly in time.

Recently Yawn and I showed that under this condition ($f_N \approx f$) the acceleration and velocity of a single particle in the SPMS can be represented by a Markov process (the *A-V* process) [15]. As expected, in the limit $N\to\infty$ the process is deterministic and the stochastic motion reduces to the Vlasov flow. We also examined the deviations from the flow. Let

$$\xi = \sqrt{N}[a(t) - a_D(t)], \quad \eta = \sqrt{N}[v(t) - v_D(t)] \tag{7}$$

where $a(t)$ and $v(t)$ are the acceleration and velocity of some test particle in the system and $a_D(t)$ and $v_D(t)$ are its image under the Vlasov flow at time t and are defined by $\xi = \eta = 0$ at the initial time $t=0$. Then, in the limit $N\to\infty$, the Markov assumption results in a Fokker-Planck equation for the probability density function (or PDF) $P(\xi,\eta;t)$,

$$\frac{\partial P(\xi,\eta;t)}{\partial t} = -\frac{\partial F_\xi P}{\partial \xi} - \frac{\partial F_\eta P}{\partial \eta} + \frac{1}{2}\frac{\partial^2 D_{\xi\xi} P}{\partial \xi^2} \tag{8}$$

with time dependent shift vector **F** and diffusion tensor **D**,

$$F_\xi = -2\eta\rho_D + \xi v_D \frac{\partial \ln \rho_D}{\partial x_D}, \quad F_\eta = \xi \tag{9}$$

$$D_{\xi\eta} = D_{\eta\xi} = D_{\eta\eta} = 0, \quad D_{\xi\xi} = 4\int_{-\infty}^{\infty} dv'\,|v' - v_D| f(x_D, v') \tag{10}$$

where, in (9), $\rho_D = \rho(x_D)$. Since **F** is linear in the state variables (ξ, η), and **D** depends only on time, the stochastic

process falls in the class known as generalized Ornstein-Uhlenbeck and has a bivariate Gaussian distribution in ξ,η [18]. In the case of equilibrium we compared the predictions of the diffusion theory with actual simulations and obtained good agreement for short times [15].

The Fokker-Planck (FP) equation can be used to obtain coupled ordinary differential equations for evolution of the moments of ξ and η which will be of use below [15]. Let $M_{m,n}(t)=\langle \xi(t)^m \eta(t)^n\rangle$ with m,n positive integers. By multiplying the FP equation, (8), by $\xi^m\eta^n$ and integrating over ξ and η, I obtain the following set of coupled ordinary differential equations for $M_{m,n}(t)$, the ensemble average of $\xi^m\eta^n$:

$$\frac{dM_{m,n}}{dt}=-2\rho_D M_{m-1,n+1}+mv_D\frac{\partial \ln\rho_D}{\partial x_D}M_{m,n} + nM_{m+1,n-1}+\frac{1}{2}m(m-1)D_{\xi\xi}M_{m-2,n}. \tag{11}$$

In particular, note that the condition $\xi(t=0)=\eta(t=0)=0$ ensures that $M_{1,0}(t)=M_{0,1}(t)=0$. In the following I will only need the second order moments $M_{0,2}(t)$, $M_{2,0}(t)$, and $M_{1,1}(t)$ which, as expected for a Gaussian process, are coupled by (11) to form a closed set.

III. DIFFUSION IN ENERGY

Consider a labeled test particle in the SPMS and identify its energy as $\varepsilon(x,v)$, the energy of the associated Vlasov orbit. As a result of diffusion, after a time t it will have position and velocity x', v' and energy $\varepsilon'=\varepsilon(x',v')$. Let $\delta x(t)$, $\delta v(t)$, and $\delta a(t)$ represent the small deviations of x, v, and a from their Vlasov images after t. The Vlasov flow provides a connection between acceleration and position so, from (6), note that

$$\delta a=\frac{\partial a}{\partial x}=-2\rho\,\delta x. \tag{12}$$

Thus the change in energy in time t can be expressed in terms of the scaled Markov variables ξ,η

$$\delta\varepsilon=v\,\delta v+\frac{a}{2\rho}\delta a=\frac{1}{\sqrt{N}}\left[v\,\delta\eta+\frac{a}{2\rho}\delta\xi\right]. \tag{13}$$

Equation (13) is important because it establishes the relationship between the energy fluctuations of a test particle and the Markovian variables ξ,η. It is easily seen from (11) that the mean vanishes ($\langle\delta\varepsilon\rangle=0$) and the variance of $\delta\varepsilon$ is given by

$$\langle(\delta\varepsilon)^2\rangle=\frac{1}{N}\left[v_D\langle(\delta\eta)^2\rangle+\frac{a_D^2}{4\rho_D^2}\langle(\delta\varepsilon)^2\rangle+\frac{a_Dv_D}{\rho_D}\langle\delta\varepsilon\,\delta\eta\rangle\right]. \tag{14}$$

A differential equation for the evolution of the variance in time is obtained by invoking (11) and recalling that on a Vlasov trajectory,

$$\dot{x}_D=v_D,\quad \dot{v}_D=a_D,\quad \dot{a}_D=\frac{\partial a_D}{\partial x_D}v_D=-\frac{\partial^2\phi}{\partial x_D^2}v_D,$$

$$\dot{\rho}_D=\frac{\partial\rho_D}{\partial x_D}v_D. \tag{15}$$

Taking the derivative of (14) wrt time then simply yields

$$\frac{d}{dt}\langle(\delta\varepsilon)^2\rangle=\frac{1}{N}\frac{a_D^2}{\rho_D^2}\int_{-\infty}^{\infty}dv'|v'-v_D|f(x_D,v'). \tag{16}$$

Note that the right-hand side of (16) depends on time only through x_D, v_D, a_D, and ρ_D which are all time periodic with period $T(\varepsilon)$. Thus the rate of increase of the variance is exactly periodic in time, so the variance after one period is independent of the initial phase of the particle on the Vlasov trajectory.

Let $\sigma_\varepsilon^2\equiv N\langle\delta\varepsilon(T)^2\rangle$ and note that it is independent of N, of time, and the initial phase of a Vlasov trajectory. For a particular choice of $f(x,v)$ it depends only on ε. Now, as N becomes large, the energy of a particle will differ very little from that of its Vlasov image after only one period. Thus, over a time scale of many periods, ε itself may be regarded as Markov. Moreover, since the change in ε during a period becomes small with increasing N, the process is continuous and can be modeled as a diffusion. As such, the Fokker-Planck equation governing $P(\varepsilon,t)$, its statistical distribution, is determined solely by $\langle\delta\varepsilon(T)\rangle/T$ and $\langle\delta\varepsilon(T)^2\rangle/T$, the first and second infinitesimal increments of $\varepsilon(T)$ [17,18]. By evaluating the increments after a time interval T, we obtain a theory which is valid on time scales where T is small. In the parlance of astrophysics literature, this is an ''orbit-averaged'' Fokker-Planck equation [20]. From the above it is clear that the first increment vanishes and the second increment is simply σ_ε^2/NT. Thus $P(\varepsilon,t)$ evolves according to the FP equation

$$\frac{\partial P(\varepsilon,t)}{\partial t}=\frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\left[\frac{\sigma_\varepsilon^2}{NT}P(\varepsilon,t)\right]. \tag{17}$$

Formally, as $N\to\infty$, $P(\varepsilon,t)$ becomes stationary. However, by introducing the scaled time $\tau=t/N$ and the diffusion ''constant'' (or function) $D(\varepsilon)=\sigma_\varepsilon^2/T$, we finally get

$$\frac{\partial P(\varepsilon,\tau)}{\partial\tau}=\frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}[D(\varepsilon)P(\varepsilon,\tau)] \tag{18}$$

for $P(\varepsilon,\tau)$ which has no explicit dependence on N.

IV. INTERPRETATION AND CONCLUSIONS

There are two possible interpretations of Eq. (18). First it provides the evolution of the energy probability density function for a labeled ''test'' particle in the system; i.e., given a particular initial energy $\varepsilon_{\rm in}$, in which case $P(\varepsilon,0)=\delta(\varepsilon-\varepsilon_{\rm in})$, its solution yields $P(\varepsilon,\tau)d\varepsilon$, the probability of finding the particle in the small energy range $d\varepsilon$ after the (scaled) time τ. The structure of the equation is deceptively simple. For an arbitrary, Vlasov stationary, $f(x,v)$, the evaluation requires the solution of (17) and the

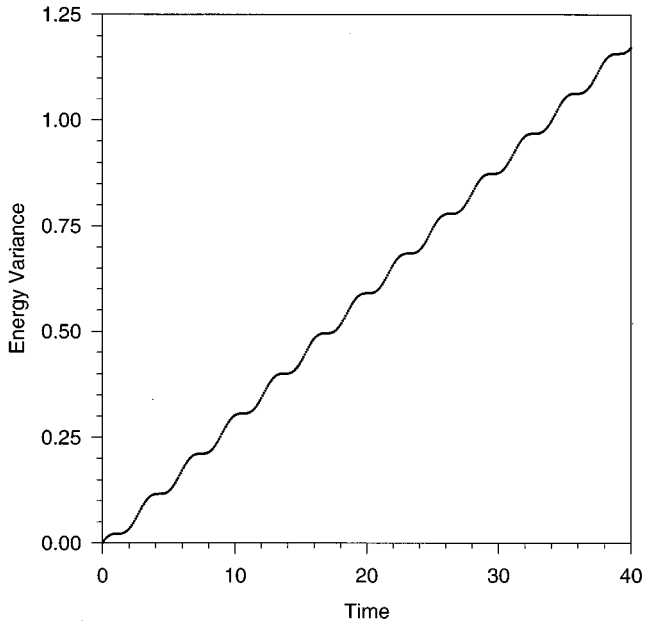

FIG. 1. Evolution of the scaled variance in energy, $N\langle \delta\varepsilon(t)^2\rangle$, versus time. Note that the behavior is nearly linear, and that the increase per period is an invariant, so that the function can be represented as a linear term plus a bounded periodic oscillation.

evaluation of $T(\varepsilon)$ for every ε [20]. In practice this can be accomplished numerically, but the details are not given here. However, let me mention that the process is simplified by taking advantage of the fact that a stationary f functionally depends on x and v only through its dependence on the specific energy $h=v^2/2+\phi(x)$ [16,21]. In Fig. 1, $N\langle \delta\varepsilon(t)^2\rangle$ is plotted for (2), the equilibrium case. Note that it increases nearly linearly with time and, more importantly, the increase per period is exactly constant, as expected. This is the key point. Thus, for a given stationary $f(x,v)$, (18) provides the microscopic relaxation of the energy distribution of a test particle resulting from its drift from orbit to orbit in μ space.

Of equal or greater importance is the fact that (18) can be used to determine the macroscopic evolution of $f(x,v)$ itself. This can be seen by noting that the energy of *each* particle in the system diffuses and therefore qualifies as a ''test'' particle. Thus $P(\varepsilon,\tau)$ can be interpreted as the energy distribution for all of the particles in the entire system. As such, it is related to $f(x,v)$ through

$$P(\varepsilon,\tau)=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dv'\,\delta[h(x',v')-\varepsilon]f(x',v';\tau)$$

$$=f(\varepsilon,\tau)/T(\varepsilon,\tau). \tag{19}$$

With this P-f connection, since D is also a functional of f [see (16) and (19)], (18) now provides a nonlinear equation for the evolution of $f(x,v)$. In practice it can be solved numerically by first advancing P for a given potential function $\phi(x)$, and then determining the change in f induced by the new P. This procedure was successfully introduced by Cohn [20] for the case of a relaxing globular cluster. In the future I plan to show that the method is also suited for the SPMSs with none of the additional approximations required in the cluster case.

The central conclusion is that, in each interpretation, time scales with N exactly and there is no additional, hidden N dependence. Thus, in agreement with the recent simulations [13], *each* relaxation process occurs on a time scale proportional to N and only differs from the other by a multiplicative constant, i.e., α or β. I will plan in a future publication to provide the details for calculating these constants.

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